

K. ADARICHEVA
M. MARÓTI
R. MCKENZIE
J.B. NATION
E.R. ZENK

The Jónsson-Kiefer property

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Abstract. The least element 0 of a finite meet semi-distributive lattice is a meet of meet-prime elements. We investigate conditions under which the least element of an algebraic, meet semi-distributive lattice is a (complete) meet of meet-prime elements. For example, this is true if the lattice has only countably many compact elements, or if $|L| < 2^{\aleph_0}$, or if \mathbf{L} is in the variety generated by a finite meet semi-distributive lattice. We give an example of an algebraic, meet semi-distributive lattice that has no meet-prime element or join-prime element. This lattice \mathbf{L} has $|\mathbf{L}| = |L_c| = 2^{\aleph_0}$ where L_c is the set of compact elements of \mathbf{L} .

Keywords: Meet semi-distributive lattice, pseudo-complemented lattice, meet-prime element, join semi-distributive lattice, join-prime element

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Introduction

In [14], B. Jónsson and J. E. Kiefer observed that in a finite join semi-distributive lattice, the canonical joinands of 1 are join-prime. The Jónsson-Kiefer property has arisen naturally in a number of settings. For example, V.A. Gorbunov and his co-researchers at Novosibirsk deeply investigated the Birkhoff-Maltsev problem, which asks for a characterization of the lattices isomorphic to the lattice $\mathbf{L}(\mathcal{K})$ of all sub-quasivarieties of \mathcal{K} for some quasivariety \mathcal{K} . Among their first discoveries was the fact that these lattices are dually algebraic lattices with the Jónsson-Kiefer property. Also, an easy argument shows that if a lattice satisfies Jónsson-Kiefer property, then it is join semi-distributive (Gorbunov [10]).

We wish to discuss various extensions and variations of the Jónsson-Kiefer property in complete lattices, and we shall identify both local and global versions of the property and its dual. For the sake of simplicity, we shall confine our discussions, whenever possible, to the dual of the property

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observed by Jónsson and Kiefer. Thus we deal primarily with meet semi-distributive lattices, with meet-prime elements, and with the dual Jónsson-Kiefer property. We will say that a complete lattice \mathbf{L} has the dual Jónsson-Kiefer property if every element a is the meet of a set of elements that are meet-prime in the interval sublattice $1/a$. Similarly, \mathbf{L} has the dual Jónsson-Kiefer property at 0 if 0 is the meet of all the meet-prime elements of \mathbf{L} . All the concepts and properties of lattices mentioned in this section are defined in the next section.

V.A. Gorbunov was concerned to show, where possible, the independence of the various properties of the lattice of sub-quasivarieties, and thus, over time he came to consider the question whether every meet semi-distributive algebraic lattice has the dual Jónsson-Kiefer property. The question seemed hard and remained open until now. It recently appeared as Problem 8 in [1]. It is resolved in the final section of this paper, where we construct an algebraic meet semi-distributive lattice that has no meet-prime element.

Concepts, terms and notation

An element c of a complete lattice \mathbf{L} is *compact* if whenever $c \leq \bigvee X$, $X \subseteq L$, then $c \leq \bigvee X'$ for some finite set $X' \subseteq X$. The set of all compact elements of \mathbf{L} will be denoted L_c . The ideal and filter generated by an element x in a lattice \mathbf{L} (or in an ordered set \mathbf{P}) are denoted $\downarrow x$ and $\uparrow x$, respectively. A lattice \mathbf{L} is termed *algebraic* if and only if \mathbf{L} is complete and for all $a \in L$, $a = \bigvee \downarrow a \cap L_c$.

A complete lattice \mathbf{L} is *upper continuous* if $a \wedge (\bigvee D) = \bigvee_{d \in D} (a \wedge d)$ for any $a \in \mathbf{L}$ and any up-directed subset $D \subseteq L$. It is *lower continuous* if the dual condition holds. For elements a, b of a lattice \mathbf{L} , we say that a is *way below* b , written $a \ll b$, if whenever U is an up-directed set and $b \leq \bigvee U$, then $a \leq u$ for some $u \in U$. A lattice is termed *Scott continuous* if for every $x \in \mathbf{L}$ we have $x = \bigvee \{a \in \mathbf{L} : a \ll x\}$. Since an element a is compact if and only if $a \ll a$, every algebraic lattice is Scott continuous. Observe that every Scott continuous lattice is upper continuous.

A lattice \mathbf{L} is termed *meet semi-distributive at a* iff

$$a = b \wedge c_1 = b \wedge c_2 \quad \text{implies} \quad a = b \wedge (c_1 \vee c_2);$$

and we say that \mathbf{L} is meet semi-distributive iff \mathbf{L} satisfies

$$u = x \wedge y = x \wedge z \quad \text{implies} \quad u = x \wedge (y \vee z).$$

Equivalently, \mathbf{L} is meet semi-distributive iff it is meet semi-distributive at each of its elements. The concept of a lattice being *join semi-distributive*

at a and the concept of *join semi-distributive lattice* are the duals of the ones just defined. We call \mathbf{L} *pseudo-complemented* if for all $x \in L$ there is a largest $y \in L$ such that $x \wedge y = 0$.

We say that $a = \bigwedge C$ *canonically* if $a = \bigwedge C$ irredundantly and whenever $a = \bigwedge B$, then C refines B in the sense that for every $c \in C$ there exists $b \in B$ such that $c \geq b$. In the context of complete lattices, we do not require B and C to be finite.

Let \mathbf{A} be a complete, upper continuous lattice, or more generally, a complete meet semilattice with 1 in which for every up-directed set U and element x , $\bigvee U = a$ exists and $x \wedge a = \bigvee \{x \wedge u : u \in U\}$. A subset S of \mathbf{A} is an *algebraic subset* if S is closed under arbitrary meets and under arbitrary joins of nonempty up-directed sets. (In particular, this requires that S contains 1.) The lattice of all algebraic subsets of \mathbf{A} ordered by set inclusion is denoted $\text{Sp}(\mathbf{A})$. One can show that $\text{Sp}(\mathbf{A})$ is join semi-distributive, and for algebraic subsets U, V in \mathbf{A} , $U \vee V$ (the join in $\text{Sp}(\mathbf{A})$) is the set $U \cup V \cup \{u \wedge v : u \in U, v \in V\}$.

Now suppose that \mathbf{L} is a complete lattice. It is not hard to show that $\text{Sp}(\mathbf{A})$ is dually algebraic iff \mathbf{L} is algebraic. A binary relation R on \mathbf{L} is said to be *distributive* if $a \wedge b R c$ implies that there exist a', b' such that $a R a'$, $b R b'$ and $c = a' \wedge b'$. A subset $S \subseteq \mathbf{L}$ is *R-closed* if $b \in S$ whenever $a \in S$ and $a R b$. For any complete lattice \mathbf{L} and any binary relation R on \mathbf{L} , the collection of all R -closed algebraic subsets of \mathbf{L} forms a complete lattice, denoted as $\text{Sp}(\mathbf{L}, R)$. The chief result on these lattices is Theorem 5 below.

Here are some basic and fairly easily established facts.

PROPOSITION 1.

(1) *If a lattice \mathbf{L} is complete and upper continuous, then it is pseudo-complemented if and only if it is meet semi-distributive at 0.*

(2) *\mathbf{L} is meet semi-distributive at a if and only*

$$a = \bigwedge B = \bigwedge C \quad \text{implies} \quad a = \bigwedge \{b \vee c : b \in B, c \in C\}$$

whenever B and C are finite subsets of \mathbf{L} .

(3) *If \mathbf{L} is finite then it is meet semi-distributive at a if and only if a has a canonical meet representation.*

(4) *If \mathbf{L} is algebraic, then it is meet semi-distributive iff for all $a, b, c \in L_c$, if $a \leq b \vee c$ then for some positive integer n and some subset $\{u_0, \dots, u_n\} \subseteq L$ with $a = u_n \geq u_{n-1} \geq \dots \geq u_0$, and some*

$\{v, v_0, \dots, v_{n-1}\} \subseteq \{b, c\}$, we have that $u_0 \leq v$ and for all $0 \leq i < n$:
 $u_{i+1} \leq u_i \vee v_i$.

An element p of a lattice \mathbf{L} is *meet-prime* if $p < 1$ and $p \geq x \wedge y$ implies $p \geq x$ or $p \geq y$. Dually, p is *join-prime* if $p > 0$ and $p \leq x \vee y$ implies $p \leq x$ or $p \leq y$. Let $\text{MP}(\mathbf{L})$ denote the set of meet-prime elements of a lattice \mathbf{L} , and let $\text{JP}(\mathbf{L})$ denote the set of its join-prime elements. Likewise, for an element $a \in \mathbf{L}$, let $\text{MP}(\uparrow a)$ denote the set of elements that are meet-prime in the filter $\uparrow a$ (i.e., elements p such that $p \geq x \wedge y \geq a$ implies $p \geq x$ or $p \geq y$), and let $\text{JP}(\downarrow a)$ denote the set of elements that are join-prime in the ideal $\downarrow a$.

We say that a complete lattice \mathbf{L} has the dual Jónsson-Kiefer property, abbreviated “dual JKP”, if $a = \bigwedge \text{MP}(\uparrow a)$ for every $a \in \mathbf{L}$. Similarly, \mathbf{L} has the dual Jónsson-Kiefer property at 0, abbreviated “dual JKP₀”, if $0_{\mathbf{L}} = \bigwedge \text{MP}(\mathbf{L})$.

The D relation on the join-irreducible elements of a lattice is defined by $p D q$ if $p \neq q$ and there exists an element $x \in \mathbf{L}$ such that $p \leq q \vee x$ but $p \not\leq r \vee x$ for all $r < q$. While this notion is most often used in finite lattices, it can also play a role in infinite lattices. The relation defined dually on the meet-irreducible elements of a lattice is denoted by D^d .

Canonical decompositions and the Jónsson-Kiefer property

The original version of Jónsson and Kiefer, appropriately dualized, reads as follows.

THEOREM 2. *If a lattice \mathbf{L} has the ascending chain condition, and is meet semi-distributive at 0, then $0_{\mathbf{L}}$ has a canonical finite decomposition into meet-prime elements.*

Gorbunov provided a natural generalization of Theorem 2 [9]. For refinements, see Semenova [16, 17] and the references therein.

THEOREM 3. *If a complete lattice \mathbf{L} is upper continuous and meet semi-distributive at 0, and every non-zero element of \mathbf{L} is above an atom, then $0_{\mathbf{L}}$ has a canonical decomposition into completely meet-prime elements.*

In a different context, standard arguments give the following. We say that the lattice \mathbf{L} is *weakly atomic* if for every $a < b$ in \mathbf{L} the interval b/a contains a two-element interval d/c . Note that every algebraic lattice is weakly atomic.

THEOREM 4. *If a complete lattice \mathbf{L} is upper continuous, weakly atomic, and distributive, then every element of \mathbf{L} is a meet of completely meet irreducible elements, which are meet-prime.*

This has nothing to do with canonical representations. Consider the lattice of open sets of a Hausdorff space, which has these properties. The coatoms (complements of singletons) are the meet-prime elements. These are generally not completely meet-prime, and the representation of 0 (say) as the meet of all meet-prime elements is not canonical.

A third result along these lines (but dually) is due to Gorbunov [10]. If \mathbf{A} is an algebraic lattice and R is a distributive relation on \mathbf{A} , then the lattice $\text{Sp}(\mathbf{A}, R)$ of all R -closed algebraic subsets of \mathbf{A} (defined in the previous section) is dually algebraic, lower continuous, and join semi-distributive.

THEOREM 5. *Let \mathbf{A} be an algebraic lattice and R a distributive relation on \mathbf{A} . Then $\text{Sp}(\mathbf{A}, R)$ has the JKP. In particular, for any quasivariety \mathcal{Q} , the lattice of subquasivarieties of \mathcal{Q} has the JKP.*

The above evidence led to the following rather optimistic extension of the Gorbunov conjecture.

CONJECTURE 6. *If \mathbf{L} is complete, upper continuous, weakly atomic and meet semi-distributive at 0, then $0 = \bigwedge \text{MP}(\mathbf{L})$.*

The natural setting for the conjecture is algebraic lattices, and in fact the lattice provided in the final section of this paper demonstrates that the conjecture is false for algebraic lattices. Before providing this counterexample, we will discuss some rather general circumstances under which meet semi-distributivity does imply the dual JKP₀.

The next result generalizes a result of M. Maróti and R. McKenzie [15]. Define an element p to be *weakly meet-prime* if $x \wedge y = 0$ implies $x \leq p$ or $y \leq p$. Let $\text{WMP}(\mathbf{L})$ denote the set of all weakly meet-prime elements of \mathbf{L} .

THEOREM 7. *A Scott continuous lattice \mathbf{L} is pseudo-complemented if and only if $\bigwedge \text{WMP}(\mathbf{L}) = 0$.*

PROOF. Suppose that $\bigwedge \text{WMP}(\mathbf{L}) = 0$ but that \mathbf{L} is not meet semi-distributive at 0, with say $0 = x \wedge y = x \wedge z < x \wedge (y \vee z) = a$. Then there exists a weakly meet-prime element p with $a \not\leq p$. Now $x \not\leq p$ as $a \not\leq p$, so $x \wedge y = 0$ implies $y \leq p$. Similarly, $x \wedge z = 0$ implies $z \leq p$. But then $a = x \wedge (y \vee z) \leq y \vee z \leq p$, a contradiction. So \mathbf{L} must be meet semi-distributive at 0. ■

Now assume that \mathbf{L} is Scott continuous and meet semi-distributive at 0. We want to show that for every element $a > 0$ there is a weakly meet-prime element p such that $a \not\leq p$. Fix $a > 0$, and choose an element b with $0 < b \ll a$. Let F be a maximal filter containing b . As observed by Gorbunov and Tumanov [13], a lattice is meet semi-distributive at 0 if and only if every maximal filter is prime. Hence the set complement of F is an ideal. Let $p = \bigvee(L - F)$.

If x, y are elements with both $x, y \not\leq p$, then $x \in F$ and $y \in F$, whence $x \wedge y \in F$ and $x \wedge y > 0$. Thus p is weakly meet-prime. On the other hand, if $a \leq p = \bigvee(L - F)$ then as $b \ll a$ we would have $b \leq \bigvee S$ for some finite subset $S \subseteq L - F$. Because $L - F$ is an ideal, that would imply $b \notin F$, a contradiction. Hence $a \not\leq p$, as desired. ■

We remarked earlier that if R is distributive and \mathbf{A} is algebraic, then $\text{Sp}(\mathbf{A}, R)$ is dually algebraic. It is also true that if R is distributive and \mathbf{A} is Scott continuous, then $\text{Sp}(\mathbf{A}, R)$ is dually Scott continuous (see, for example, Adaricheva [2] for the case of $R = id$).

The importance of these ideas is due to the fact that for any quasivariety \mathcal{Q} , the lattice $\mathbf{L}_{\mathcal{Q}}$ of all subquasivarieties of \mathcal{Q} can be represented as $\text{Sp}(\mathbf{A}, R)$ for an algebraic lattice \mathbf{A} and a distributive relation R (Gorbunov and Tumanov [12]; see also [11]). Gorbunov's proof of the JKP for these lattices is part of the effort to characterize the lattices $\mathbf{L}_{\mathcal{Q}}$, with the aid of some strong form of join semi-distributivity. The lattices $\text{Sp}(\mathbf{A}, R)$ are also useful as a source of examples of join semi-distributive lattices.

Gorbunov's result (Theorem 5) can be generalized to the case when the lattice \mathbf{A} is Scott continuous.

THEOREM 8. *Let \mathbf{A} be a Scott continuous lattice and R a distributive relation on \mathbf{A} . Then $\text{Sp}(\mathbf{A}, R)$ has the JKP.*

PROOF. First, it is folklore that any algebraic subset of a Scott continuous lattice is also a Scott continuous lattice (with respect to the induced order). To see this, let S be an algebraic subset of \mathbf{A} and take $a \in S$. Then $a = \bigvee b_i$ with $b_i \in \mathbf{A}$ and $b_i \ll a$. Consider $a_i = \bigwedge \{x \in S : x \geq b_i\}$. Clearly $a_i \in S$ and $b_i \leq a_i \leq a$ for each i , so $a = \bigvee a_i$. Fixing an index i , suppose $a \leq \bigvee' D$ where D is an up-directed subset of S and \bigvee' denotes the join in S . Because S is algebraic, $\bigvee' D = \bigvee D$ and hence $b_i \leq d$ for some $d \in D$. Then $a_i \leq d$, and we have shown that $a_i \ll a$ in the lattice S . Thus S is Scott continuous.

Also recall that every element in a Scott continuous lattice is a meet of meet irreducible elements [7].

With these two facts, the proof just repeats the one in Gorbunov [10], Lemma 6.6. ■

In the next section, we show that in any algebraic pseudo-complemented lattice \mathbf{L} with $|\mathbf{L}| < 2^{\aleph_0}$, $\bigwedge \text{MP}(\mathbf{L}) = 0$. In Section 5 it is proved that any algebraic meet semi-distributive lattice in which $|\mathbf{L}_c| \leq \aleph_0$ or \mathbf{L}_c is a sublattice satisfies $\bigwedge \text{MP}(\mathbf{L}) = 0$. We also show that every atomistic dually algebraic lattice that supports an equaclosure operator has the JKP. In Section 6, we show that $\bigwedge \text{MP}(\mathbf{L}) = 0$ for any weakly atomic, join continuous member \mathbf{L} of a variety generated by a finite meet semi-distributive lattice.

Using the structure of pseudo-complemented lattices

We recall a basic structure result for pseudo-complemented complete lattices (Glivenko [8], Frink [6]).

THEOREM 9. *Let \mathbf{L} be a pseudo-complemented complete lattice, and let $\mathbf{L}^* = \{x^* : x \in \mathbf{L}\}$. Then \mathbf{L}^* is a complete Boolean algebra. The meet operation on \mathbf{L}^* coincides with that on \mathbf{L} , and the join in \mathbf{L}^* of a subset $X \subseteq \mathbf{L}^*$ is given by $(\bigwedge \{x^* : x \in X\})^*$. The complementation on \mathbf{L}^* is given by $*$.*

Moreover, the closure operator $\gamma : \mathbf{L} \rightarrow \mathbf{L}^$ via $\gamma(x) = x^{**}$ is an increasing, 0-separating homomorphism that preserves finite meets and arbitrary joins.*

This applies immediately to the current problem.

THEOREM 10. *Let \mathbf{L} be a pseudo-complemented complete lattice. If \mathbf{L}^* is atomic, then 0 is the meet of the meet-prime elements of \mathbf{L} . If 0 has a canonical meet representation, then \mathbf{L}^* is atomic.*

PROOF. Assume that \mathbf{L}^* is atomic, and let c be a coatom of \mathbf{L}^* . Suppose that c is not meet-prime, say $c \geq x \wedge y$ properly. Now $c^* \wedge x > 0$ since $x \not\leq c^{**} = c$, and $(c^* \wedge x) \wedge y \leq c^* \wedge c = 0$. Thus $y \leq (c^* \wedge x)^*$, so that $c < (c^* \wedge x)^* < 1$, a contradiction. Hence the coatoms of \mathbf{L}^* are meet-prime.

Assume that $0 = \bigwedge X$ canonically in \mathbf{L} , and let $x \in X$. Let $m = \bigwedge (X - \{x\})$, so that by the definition of a canonical meet representation $x^* = m$ and $m^* = x$. If $1 > z^* \geq x$ for some element $z \in \mathbf{L}$, then $z \leq z^{**} \leq x^* = m$. Since $z > 0$ we have $z \not\leq x$, and thus $z \wedge z^* = 0$ implies $z^* \leq x$. Thus x is a coatom of \mathbf{L}^* . ■

Now recall that every infinite complete Boolean algebra has cardinality at least 2^{\aleph_0} . The next result is thus a consequence of Theorems 9 and 10.

COROLLARY 11. *Let \mathbf{L} be a pseudo-complemented complete lattice with $|\mathbf{L}| < 2^{\aleph_0}$. Then \mathbf{L}^* is finite, and hence 0 is the meet of the meet-prime elements of \mathbf{L} .*

We now show that the statement of Theorem 10 cannot be strengthened to an equivalence by giving an example of a complete (algebraic) pseudo-complemented lattice in which the pseudo-complements form an atomless Boolean algebra and the meet of all meet-prime elements is 0.

Let C denote the Cantor space (the product of countably many discrete two-point spaces with the product topology). The lattice $\mathcal{O}(C)$ of open sets of the Cantor space has the desired properties. Clearly $\mathcal{O}(C)$ is complete; it is even algebraic (because C is zero-dimensional and compact). It is pseudo-complemented because of the frame distributive law: $(\bigcup U_i) \cap V = \bigcup (V \cap U_i)$. Indeed, for any open set U , the pseudo-complement U^* is the interior of the set complement $C - U$. Thus $U = U^{**}$ if and only if U is regular open. (A set U is *regular open* if U is the interior of the closure of U .) Let \mathbf{B} denote the lattice of regular open sets of C . Then \mathbf{B} is atomless, because any pair of points can be separated by a regular open set (in fact, by a clopen set). As C has no isolated points, \mathbf{B} is atomless. Finally, for each point $x \in C$ the set complement $C - \{x\}$ is a meet-prime open set. The meet of all such prime open sets clearly contains no point of C .

We conclude this section with some observations about the meet-prime elements in a pseudo-complemented complete lattice.

THEOREM 12. *Let \mathbf{L} be a pseudo-complemented complete lattice.*

1. *If p is a meet-prime element of \mathbf{L} , then p^{**} is either 1 or a coatom of \mathbf{L}^**
2. *If $0 = \bigwedge \text{MP}(\mathbf{L})$, then $b = \bigwedge (\uparrow b \cap \text{MP}(\mathbf{L}))$ for every $b \in \mathbf{L}^*$.*

PROOF. If p is meet-prime in \mathbf{L} , then clearly for every $a \in \mathbf{L}$ either $p \geq a$ or $p \geq a^*$. Suppose $p^{**} < a^* < 1$. Then $a \wedge p^{**} = 0$, whence we cannot have $a \leq p \leq p^{**}$, while $p^{**} \geq p \geq a^* > p^{**}$ is also a contradiction. Thus p^{**} must be either 1 or a coatom of \mathbf{L}^* .

Assuming that $0 = \bigwedge \text{MP}(\mathbf{L})$, for $a \in \mathbf{L}$ let $\mu(a) = \bigwedge (\uparrow a \cap \text{MP}(\mathbf{L}))$. Then $a \leq \mu(a)$ and $\mu(a) \wedge \mu(a^*) = 0$, whence $\mu(a^*) \leq (\mu(a))^* \leq a^* \leq \mu(a^*)$. Thus $\mu(a^*) = a^*$. ■

Sufficient conditions for the dual JKP in algebraic lattices

Clearly some finiteness condition, in addition to join (or meet) semi-distributivity, ■ needs to be imposed in order to ensure that a lattice has the JKP or its dual.

As observed by Gorbunov [10], the lattice $\mathbf{Co}(\mathbf{Z})$ of convex subsets of the integers is join semi-distributive, algebraic and atomistic, yet has no join-prime element.

Recall that a filter \mathcal{F} is *prime* if $a, b \notin \mathcal{F}$ implies $a \vee b \notin \mathcal{F}$. Equivalently, a filter \mathcal{F} is prime if and only if $\mathbf{L} - \mathcal{F}$ is an ideal.

LEMMA 13. *Let \mathbf{L} be an algebraic lattice. The following are equivalent.*

1. *For every compact element $b > 0$, there exists a prime filter \mathcal{F} such that $b \in \mathcal{F}$ and $\mathcal{F} = \uparrow(\mathcal{F} \cap \mathbf{L}_c)$, that is, every element of \mathcal{F} is above a compact element in \mathcal{F} .*
2. $0 = \bigwedge \text{MP}(\mathbf{L})$.

PROOF. Assume (1), and suppose that $u = \bigwedge \text{MP}(\mathbf{L}) > 0$. Then there is a compact element b such that $0 < b \leq u$. Let \mathcal{F} be as given by (1), and let $\mathcal{J} = \mathbf{L} - \mathcal{F}$. As \mathcal{F} is prime, \mathcal{J} is an ideal. So \mathcal{J} is closed under finite joins, and by the second property in (1) we see that $p = \bigvee \mathcal{J}$ is not in \mathcal{F} . Thus $\mathcal{J} = \downarrow p$. Since $\mathbf{L} - \downarrow p = \mathcal{F}$ is a filter, p is meet-prime. But $p \not\leq b$ since $b \in \mathcal{F}$, and hence $p \not\leq u$, contradicting the definition of u .

Conversely, assume that $0 = \bigwedge \text{MP}(\mathbf{L})$ and let $b \in \mathbf{L}_c$ with $b > 0$. Then there is a meet-prime element p such that $p \not\leq b$. Let $\mathcal{J} = \downarrow p$ and $\mathcal{F} = \mathbf{L} - \mathcal{J}$, which will be a prime filter. For any $x \in \mathcal{F}$ we have $x \not\leq p$, whence there is a compact element c with $c \leq x$ and $c \not\leq p$. Then $c \in \mathcal{F}$, and we have shown that $\mathcal{F} = \uparrow(\mathcal{F} \cap \mathbf{L}_c)$. ■

Gorbunov and Tumanov [13] also observe that a lattice \mathbf{L} is meet semi-distributive at 0 if and only if every maximal filter of \mathbf{L} is prime. Thus in any lattice with 0 that is meet semi-distributive at 0, every element $x > 0$ is contained in a prime filter. It follows that the ideal lattice $\mathcal{J}(\mathbf{L})$ of a lattice that is meet semi-distributive at 0 satisfies $0 = \bigwedge \text{MP}(\mathcal{J}(\mathbf{L}))$. (This despite the fact that the ideal lattice of a meet semi-distributive lattice need not be meet semi-distributive.) Also in [13], there is an example of a lattice satisfying both semi-distributive laws and Whitman's condition (W), but having no least element and no prime filter. In the notes [4], there is an example of a meet semi-distributive lattice \mathbf{L} that satisfies the ACC (and hence is algebraic) and has $\mathbf{L} - \{0\}$ as its *only* prime filter.

We will use a slight weakening of maximality for filters.

LEMMA 14. *Let \mathbf{L} be an algebraic lattice that is meet semi-distributive at 0. If \mathcal{F} is a filter of \mathbf{L} such that*

- (i) $\mathcal{F} = \uparrow(\mathcal{F} \cap \mathbf{L}_c)$, and

(ii) for every compact element $x \notin \mathcal{F}$, there exists $y \in \mathcal{F}$ such that $x \wedge y = 0$, then \mathcal{F} is prime.

PROOF. Assume that \mathcal{F} satisfies (i) and (ii). Suppose $a, b \notin \mathcal{F}$ but $a \vee b \in \mathcal{F}$. Then $a \vee b \geq f$ for some compact $f \in \mathcal{F}$, whence by (i) there exist compact elements $a' \leq a$ and $b' \leq b$ with $a' \vee b' \geq f$. On the other hand, by (ii), $a' \notin \mathcal{F}$ implies that $a' \wedge g = 0$ for some $g \in \mathcal{F}$. Likewise, $b' \wedge h = 0$ for $h \in \mathcal{F}$. Replacing f, g and h with their meet, we may assume that $f = g = h$. By the meet semi-distributivity at 0, we have $0 = (a' \vee b') \wedge f = f$, a contradiction. Thus \mathcal{F} is prime. ■

THEOREM 15. *Let \mathbf{L} be an algebraic lattice that is meet semi-distributive at 0. If \mathbf{L}_c is a sublattice of \mathbf{L} , then $0 = \bigwedge \text{MP}(\mathbf{L})$.*

PROOF. Given $b > 0$ compact, then using Zorn's Lemma let \mathcal{F} be a filter that is maximal such that $b \in \mathcal{F}$, $0 \notin \mathcal{F}$, and $\mathcal{F} = \uparrow(\mathcal{F} \cap \mathbf{L}_c)$. If x is compact and $x \notin \mathcal{F}$, then the filter \mathcal{G} generated by $\mathcal{F} \cup \{x\}$ has the property that $\mathcal{G} = \uparrow(\mathcal{G} \cap \mathbf{L}_c)$, whence $0 \in \mathcal{G}$, as desired. ■

THEOREM 16. *Let \mathbf{L} be an algebraic lattice that is meet semi-distributive at 0. If \mathbf{L}_c is countable, then $0 = \bigwedge \text{MP}(\mathbf{L})$.*

PROOF. Let $b > 0$ be a compact element. Index $\mathbf{L}_c = \{b_0, b_1, b_2, \dots\}$ with $b_0 = b$. Let $f_0 = b_0$.

Inductively, suppose we have compact elements $f_0 \geq f_1 \geq \dots \geq f_k > 0$ such that for all $j \leq k$, either $b_j \geq f_k$ or $b_j \wedge f_k = 0$. If $b_{k+1} \wedge f_k = 0$, put $f_{k+1} = f_k$. If $b_{k+1} \wedge f_k > 0$, let f_{k+1} be a nonzero compact element below $b_{k+1} \wedge f_k$. Note that the property now holds for $k + 1$.

Let $\mathcal{F} = \bigcup_{k \in \omega} \uparrow f_k$. By Lemma 14 this is prime, whence by Lemma 13, $0 = \bigwedge \text{M}(\mathbf{L})$. ■

However, we note that even in a countable, distributive, algebraic lattice, it is possible to have a maximal filter \mathcal{F} such that $\mathcal{F} \neq \uparrow(\mathcal{F} \cap \mathbf{L}_c)$, so that its complement is not a principal ideal.

Next, we examine the implications of having an equaclosure operator (a property of the lattices of sub-quasivarieties). Let \mathbf{L} be a complete lattice with a closure operator $h : \mathbf{L} \rightarrow \mathbf{L}$ that satisfies properties (h1)–(h4) of equaclosure operators (see [3]; also[11], p.195):

$$(h1) \quad h(0) = 0.$$

$$(h2) \quad \text{For all } x, y \in L, h(x) = h(y) \text{ implies } h(x) = h(x \wedge y).$$

(h3) For all $x, y, z \in L$, $h(x) \wedge (y \vee z) = (h(x) \wedge y) \vee (h(x) \wedge z)$.

(h4) $h(L)$ with the induced order is a dually algebraic lattice and any $x \in h(L)$ is co-compact in $h(L)$ iff x is co-compact in \mathbf{L} .

We call a lattice *atomistic* if every element is a join of atoms.

THEOREM 17. *Let \mathbf{L} be a dually algebraic lattice that admits an equaclosure operator h .*

(1) $h(\bigvee JP(\mathbf{L})) = 1_{\mathbf{L}}$.

(2) *If \mathbf{L} is atomistic then \mathbf{L} has the JKP.*

PROOF. Let \mathbf{L} be a dually algebraic lattice with an equaclosure operator h . For (1), it suffices to show that for each co-compact element $c \in h(L)$ with $c < 1$ there is a join-prime element of \mathbf{L} not below c . We begin by choosing s to be a minimal element in $h(L) \setminus \downarrow c$, which we can do because c is co-compact in $h(L)$. The dual algebraicity of $h(L)$ easily yields that (h2) and (h4) imply the existence of the least element $e \in L$ with the property $h(e) = s$. Now $e \not\leq c$, and we can show that e is join-prime in \mathbf{L} as follows. Suppose that $a \in L$, $a \not\leq e$. Then $h(a \wedge s) < s$ (else $s = h(e) = h(a \wedge s) = h(e \wedge a \wedge s)$), and thus $a \wedge s \leq h(a \wedge s) \leq c$. Suppose also that $b \in L$, $b \not\leq e$, so that also $h(b \wedge s) \leq c$. Then by (h3),

$$(a \vee b) \wedge s = (a \wedge s) \vee (b \wedge s) \leq c.$$

Thus $h((a \vee b) \wedge s) \leq c$, implying that $a \vee b \not\leq e$. We have shown that $e \in JP(\mathbf{L})$, as required.

For (2), we pick any dually compact element $d < 1$ in L . In order to show that L has JKP at 1 it will suffice to find a join-prime atom not below d . Let's assume to the contrary that no atom in L belonging to $F = L \setminus \downarrow d$ is join-prime. Pick any atom $x \in F$. There are elements $u, v \in L$ such that $x \leq u \vee v$, but $x \not\leq u, v$. Now $x \leq h(x) \wedge (u \vee v) = (h(x) \wedge u) \vee (h(x) \wedge v)$, so $h(x) \wedge u$ and $h(x) \wedge v$ cannot both be below d . Hence there must be an atom $y \leq h(x) \wedge u$ or $y \leq h(x) \wedge v$ with $y \in F$. Now $h(y) \leq h(x)$ and $x \wedge y = 0$, whence by (h2) we conclude that $h(y) < h(x)$.

Now let $S = \{h(x) : x \in F \text{ is an atom}\}$. Let M be a maximal chain in S , and put $m = \bigwedge M$. Since d is co-compact, then $m \in F$. We can choose an atom $a \leq m$, $a \in F$. Now $h(a) \leq m$ and, as above there must exist some atom $b \in F$ with $h(b) < h(a)$. The chain $M \cup \{h(b)\}$ contradicts the maximality of M . This contradiction proves the desired result, that there is a join-prime atom in F .

Finally, to show that L has JKP we mention that if L admits an equaclosure operator h then every principal ideal $\downarrow a$ of L admits the equaclosure operator $h_a(x) = h(x) \wedge a$, see [3]. ■

Varieties with the dual JKP

First, we prove a generalization of the result for distributive lattices.

THEOREM 18. *Let \mathbf{L} be a finite meet semi-distributive lattice. If \mathbf{K} is a complete, upper continuous, weakly atomic lattice in $\mathcal{V}(\mathbf{L})$, then $0 = \bigwedge \text{MP}(\mathbf{K})$.*

PROOF. We need to show that if $u > 0$ in \mathbf{K} , then there is a meet-prime element not above u . By weak atomicity, there is a covering pair in \mathbf{K} with $a < b \leq u$. We will find a meet-prime element not above b .

Let ψ be the unique maximum congruence of \mathbf{K} separating a and b . Then \mathbf{K}/ψ is a subdirectly irreducible lattice in $\mathcal{V}(\mathbf{L})$, and hence a finite, meet semi-distributive lattice. Let us show that the natural map $\phi : \mathbf{K} \rightarrow \mathbf{K}/\psi$ is upper bounded. To see that this will suffice, note that the canonical meetands of 0 in \mathbf{K}/ψ are meet-prime, and the greatest preimage of a meet-prime element is meet-prime. Since $\phi(b) > 0$, it follows that the greatest preimage of at least one of the canonical meetands of 0 is not above b .

By upper continuity, we can find a maximal element p above a and not above b . (By SD_\wedge , p is unique, but this part of the argument works without using the uniqueness of p .) Now p is completely meet irreducible, and $[\phi(p), \phi(p^\dagger)]$ is a critical quotient in \mathbf{K}/ψ , where p^\dagger is the unique upper cover of p . Moreover, p is the greatest preimage of $\phi(p)$. For every meet irreducible element $d \in \text{M}(\mathbf{K}/\psi)$, we have

$$\phi(p) \text{ D}^d c_1 \text{ D}^d \dots \text{ D}^d c_{k-1} \text{ D}^d d$$

for some $k \geq 0$. (See e.g. Corollary 2.37 of [5].) It suffices to prove that the meet irreducible elements have greatest preimages, so we will prove that if $c \text{ D}^d d$ and c has a greatest preimage, then so does d .

Assume that $c \geq d \wedge x$ properly in \mathbf{K}/ψ , and $c \not\geq d^\dagger \wedge x$, and that c has a greatest preimage q in \mathbf{K} . Choose any $t \in \phi^{-1}(x)$. Then $c \geq \phi(s \wedge t)$ for all $s \in \phi^{-1}(d)$, so that $q \geq \bigvee_{\phi(s)=d} (s \wedge t) = (\bigvee \phi^{-1}(d)) \wedge t$ by upper continuity. Hence $\bigvee \phi^{-1}(d)$ is a preimage of d , necessarily the greatest one. This completes the proof. ■

A counterexample

The lattice \mathbf{L} constructed in this section is algebraic, meet semi-distributive, has compact 1, is of cardinality continuum, and has no meet-prime elements or join-prime elements. The most remarkable thing about \mathbf{L} is that it is easily described and accessible to inspection.

We require more definitions. Let $\mathbf{P} = \langle P, \leq \rangle$ be an ordered set (i.e., a partially ordered set). A *condition* for \mathbf{P} is a triple $c = (a, u, v)$ of elements of P with u, v incomparable, $a \not\leq u, a \not\leq v$. An order ideal (downset) J in \mathbf{P} is said to *satisfy* the condition c iff $\{u, v\} \subseteq J \rightarrow a \in J$. Let C be a set of conditions for \mathbf{P} . We have the algebraic lattice $\mathbf{L}(\mathbf{P}, C)$ whose elements are the downsets of \mathbf{P} that satisfy all the conditions in C . The order in $\mathbf{L}(\mathbf{P}, C)$ is set inclusion.

Let S^{ω} be the infinite binary-branching rooted tree consisting of all functions t such that $\text{dom}(t)$ is some natural number $n = \{0, 1, \dots, n-1\}$ and $t : n \rightarrow \{0, 1\}$. We order S^{ω} by putting $s \leq t$ iff $s \supseteq t$. The largest element of this ordered set is the empty function \emptyset . Note that $\langle S^{\omega}, \leq \rangle$ is actually an upper semilattice, the join of σ and τ being $\sigma \cap \tau$.

Define Σ as the subset of $S^{\omega} \times S^{\omega}$ consisting of all pairs (t_0, t_1) with $t_0 \leq t_1$ (i.e., with $t_1 \subseteq t_0$), and give Σ the the product order, so that $(s_0, s_1) \leq (t_0, t_1)$ iff $s_0 \leq t_0$ and $s_1 \leq t_1$. For $s \in S^{\omega}$ we write $|s|$ for the length of s , i.e., the domain of s . For $t = (t_0, t_1) \in \Sigma$, we define $\sigma(t) = t_0$, $n(t) = |t_1|$. Notice that for $s, t \in \Sigma$ we have $s \leq t$ iff $\sigma(s) \leq \sigma(t)$ and $n(s) \geq n(t)$.

We write Π for the set of all maximal chains in $\langle S^{\omega}, \leq \rangle$. The members of Π will often be called *paths* in S^{ω} . Let $s : \Pi \rightarrow P_{\pi}$ be a bijection between Π and a set $P_{\pi} = \{s_p : p \in \Pi\}$, which we assume to be disjoint from Σ . We take

$$P = \Sigma \cup P_{\pi},$$

and order P as follows. For $s = (s_0, s_1), t = (t_0, t_1) \in \Sigma$, and for $p \in \Pi$ we have

- $s \leq t$ means that $s \leq t$ in Σ .
- $s < s_p$ iff $s_1 \in S^{\omega} \setminus p$ (and so $s_0 \in S^{\omega} \setminus p$).

In this order, there is no top element, and the maximal elements of P are the members of P_{π} and the top element, (\emptyset, \emptyset) , of Σ . We note that if $p \in \Pi$

and $t = (t_0, t_1) \in \Sigma$, then t is a maximal member of $\downarrow s_p \setminus \{p\}$ iff $t_0 = t_1 \notin p$ and for all $s > t$, $s \in S^\omega$ we have $s \in p$.

We require the following definitions. For $s, s' \in S^\omega$ we write $s' \prec s$ to denote that $s \subseteq s'$ and $|s'| = |s| + 1$, in other words, s' is one of the two subcovers, $s0$ and $s1$, of s in the ordered set S^ω . Let $t = (t_0, t_1)$, $t' = (t'_0, t'_1) \in \Sigma$. Then we shall write $t \triangleright t'$ iff $t_1 \geq t'_1$ in Σ ; $t \sim_0 t'$ iff $t_0 = t'_0$; $t \sim_1 t'$ iff $t_1 = t'_1$.

Note that $x \triangleright y$ implies $x \triangleright y'$ for all $y' \leq y$; also, if $x \sim_1 x'$ and $y \sim_1 y'$ then $x \triangleright y$ iff $x' \triangleright y'$. Note also that for $p \in \Pi$, our definition gives $x < s_p$ iff $x \in \Sigma$ and for no s in p do we have $x \triangleright (s, s)$.

We define a set C of conditions on the ordered set $\mathbf{P} = \langle P, \leq \rangle$. It is the union of four sets C_i ($0 \leq i \leq 3$).

C_0 : the set of all triples $((s, t), (s0, t), (s1, t))$ with $s, t \in S^\omega$, $s \leq t$.

C_1 : the set of all triples $((s, t), (s, u), (r, t))$ with $\{r, s, t, u\} \subseteq S^\omega$, $s \leq t$, $s \leq u$, $r \leq t$.

C_2 : the set of all triples $((s, t), (si, t), s_p)$ with $p \in \Pi$, and $s \leq t$, where $si \prec s$, $s \in p$, $si \notin p$.

C_3 : the set of all triples $((s_q, s_p, (ti, ti))$ where $\{p, q\} \subseteq \Pi$, $p \neq q$, $i \in \{0, 1\}$, $t \in p \cap q$, and $ti \in p \setminus q$.

Finally, we take \mathbf{L} as the lattice $\mathbf{L}(\mathbf{P}, C)$. In other words, \mathbf{L} is the set of all downsets J in \mathbf{P} such that for all $(x, y, z) \in C$, if $\{y, z\} \subseteq J$ then $x \in J$, and it is ordered by set inclusion. For $Y \subseteq P$, we shall write \overline{Y} for the smallest member of \mathbf{L} containing the set Y .

We note that $\Sigma \in L$, and for $Y \subseteq \Sigma$, \overline{Y} is the smallest downset $J \subseteq \Sigma$ such that J satisfies conditions $C_0 \cup C_1$. In order to prove that \mathbf{L} is meet semi-distributive, we need a more useful characterization of \overline{Y} for $Y \subseteq \Sigma$. Thus for $Y \subseteq \Sigma$ we define Y^\dagger be the smallest set T containing Y such that for all $(x, y, z) \in C_0$, if $\{y, z\} \subseteq T$ then $x \in T$. We define Y^\triangleright to be the set of all $x \in \Sigma$ such that $y \triangleright x \sim_0 z$ for some $\{y, z\} \subseteq (\downarrow Y)$.

LEMMA 19.

(1) For $x \in P$, $\downarrow x$ is a member of \mathbf{L} . Thus \mathbf{P} is naturally embedded in \mathbf{L} .

(2) If $\{p, p', q\} \subseteq \Pi$, $p \neq p'$, and $p \cap p' \subseteq q$, then $s_q \in \overline{\{s_p, s_{p'}\}}$.

- (3) If $Y \subseteq P$ and $\{y, z\} \subseteq \downarrow Y$ and $y \triangleright x \sim_0 z$, then $x \in \overline{Y}$.
- (4) If $Y \subseteq \Sigma$ then \overline{Y} is the set of all $x \in \Sigma$ satisfying (ℓ_x^Y) : there is a finite set $Q_x \subseteq \downarrow x$ such that $x \in Q_x^\dagger$, $Q_x \subseteq Y^\triangleright$, and for each $q \in Q_x$, $x \sim_1 q$.

PROOF. Statement (1) is straightforward to verify. For (2), assume that $\{p, p', q\} \subseteq \Pi$, $p \neq p'$, and $p \cap p' \subseteq q$. Let $s \in S^\omega$ be the least member of $p \cap p'$ and $s' \prec \sigma$ be the subcover of s that lies off q . Now s' belongs to one but not both of p, p' . We can assume, without losing generality, that $s' \in p \setminus p'$. Then we have that $x = (s', s') \leq s_{p'}$ and, by C_3 , $s_q \in \overline{\{s_p, x\}}$. Thus $s_q \in \overline{\{s_p, s_{p'}\}}$.

Statement (3) follows immediately on consideration of the conditions C_1 . Now to prove (4), let $Y \subseteq \Sigma$. It is clear that \overline{Y} is the smallest downset contained in Σ which contains Y and satisfies conditions $C_0 \cup C_1$. Define $\partial(Y)$ as the set of all $x \in \Sigma$ such that (ℓ_x^Y) holds. It is easily verified that

$$Y \subseteq Y^\triangleright \subseteq \partial(Y) \subseteq \overline{Y}$$

and that Y^\triangleright is a downset. We need to prove that $\partial(Y)$ is a downset that satisfies the conditions of $C_0 \cup C_1$.

First, suppose that $x = (s, t) \in \Sigma$ and where $x_i = (s_i, t)$ ($i \in \{0, 1\}$) we have $\downarrow x_i \subseteq \partial(Y)$ for $i \in \{0, 1\}$. We show that $\downarrow x \subseteq \partial(Y)$. Let $y = (u, v) \leq x$. If $u < s$ then $y \leq x_i$ ($i = 0$ or $i = 1$) and so $y \in \partial(Y)$. So we can assume that $y = (s, v)$, $v \leq t$. Now $y_i \leq x_i$, so $y_i = (s_i, v) \in \partial(Y)$ ($i \in \{0, 1\}$); hence we have $(\ell_{y_i}^Y)$. Let $Q_i \subseteq \downarrow y_i \cap Y^\triangleright$, so that $y_i \in Q_i^\dagger$ and for each $q \in Q = Q_0 \cup Q_1$, $q \sim_1 y$ ($y \sim_1 y_0 \sim_1 y_1$). Now we have $y \in Q^\dagger$, obviously (use conditions C_0), and we have shown that $y \in \partial(Y)$.

From the argument of the preceding paragraph, it is clear that $\partial(Y)$ is a downset and satisfies C_0 .

To see that $\partial(Y)$ satisfies C_1 , let $x = (s, t)$, $y = (s, u)$, $z = (r, t)$ with $s \leq t$, $s \leq u$ and $r \leq t$, and with $\{y, z\} \subseteq \partial(Y)$. We need to prove that $x \in \partial(Y)$. We assume that $t > u$ (else $x \leq y$ and there is nothing to prove). We have that (ℓ_y^Y) and (ℓ_z^Y) hold. We have only to show that (ℓ_x^Y) holds.

Choose $Q_y \subseteq \downarrow y \cap Y^\triangleright$ and $Q_z \subseteq \downarrow z \cap Y^\triangleright$ so that $y \in Q_y^\dagger$, $z \in Q_z^\dagger$ and $y \sim_1 q$ for all $q \in Q_y$, and similarly for Q_z . Define $Q_x = \{(w, t) : (w, u) \in Q_y\}$. (Since $s \leq u < t$ then $(w, t) \in P$ whenever $(w, u) \in P$.) Clearly, $Q_x \subseteq \downarrow x$ and $x = (s, t) \in Q_x^\dagger$ as $y = (s, u) \in Q_y^\dagger$. We must show that $Q_x \subseteq Y^\triangleright$. So choose $\bar{x} = (a, t) \in Q_x$. Put $\bar{y} = (a, u) (\in Q_y)$ and choose $\bar{z} = (b, t) \in Q_z$.

Choose $\beta \in \downarrow Y$ with $\beta \triangleright \bar{z}$ and choose $\alpha \in \downarrow Y$ with $\alpha \sim_0 \bar{y} \sim_0 \bar{x}$. Since $\beta \triangleright \bar{z}$ and $\bar{z} \sim_1 \bar{x}$ then $\beta \triangleright \bar{x}$. Now $\{\alpha, \beta\} \subseteq \downarrow Y$ and $\beta \triangleright \bar{x} \sim_0 \alpha$. This shows that $\bar{x} \in Y^\triangleright$. Thus we have shown that (ℓ_x^Y) holds, and $x \in \partial(Y)$ as desired. This concludes our proof of (4). \blacksquare

We define some subsets of P . Let $n \in \omega$, $p \in \Pi$ and $\sigma \in S^\smile$.

$$\begin{aligned}\Sigma_n &= \{t \in \Sigma : n(t) = n\}; \\ \Sigma_{\geq n} &= \{t \in \Sigma : n(t) \geq n\}; \\ S_p &= \{x \in P : x \leq s_p\};\end{aligned}$$

$$\begin{aligned}S_\sigma &= \{s_q \in P_\pi : \sigma \in q\} \cup \{x \in \Sigma : \neg\{x \triangleright (\sigma, \sigma)\}\}. \\ T_p &= \{(s, t) \in \Sigma : s \notin p\}.\end{aligned}$$

Notice that for $p \in \Pi$, $S_p \setminus \{s_p\}$ is the set of all $x \in \Sigma$ such that for all $s \in p$, $\neg\{x \triangleright (s, s)\}$. From this, it follows that

$$S_p = \bigcap_{\sigma \in p} S_\sigma.$$

We can observe that Σ_n (called the n -level in Σ) as an ordered subset of Σ is isomorphic to the disjoint union of 2^n copies of the ordered set S^\smile , and $\Sigma_{\geq n} = \downarrow \Sigma_n$ is isomorphic to the disjoint union of 2^n copies of Σ , namely the sets $\downarrow(s, s)$ with $s \in S^\smile$, $|s| = n$. To easily understand the proofs that follow, it helps to observe that for $x \in \Sigma_n$ and $y \in \Sigma$, we have $x \triangleright y$ iff y belongs to the same connected component of $\Sigma_{\geq n}$ as x .

LEMMA 20.

- (i) We have $\Sigma = \Sigma_{\geq 0} = \downarrow(\emptyset, \emptyset)$; and for all $n \in \omega$, $p \in \Pi$ and $\sigma \in \Sigma$, the sets $\Sigma_{\geq n}$, S_p , S_σ and T_p are members of L .
- (ii) For each $p \in \Pi$, we have $\mathbf{L} \models S_p \vee T_p = 1_L$ and $\mathbf{L} \models S_p = \bigwedge_{\sigma \in p} S_\sigma$.
- (iii) For each $\{p, q\} \subseteq \Pi$, $p \neq q$ implies $S_p \wedge S_q \leq \Sigma$ and $S_p \vee S_q = S_\sigma$ where σ is the least element of $p \cap q$.
- (iv) \mathbf{L} has no meet-prime elements and no join-prime elements.

PROOF. The verification of (i) is left to the reader. For (ii), we have already observed that $S_p = \bigcap_{\sigma \in p} S_\sigma$. Now let $p \in \Pi$ and let $i \prec \emptyset$ be the subcover of \emptyset in S^\smile that lies outside p . Then $(i, \emptyset) \in T_p$, and with conditions C_2 it follows that $(\emptyset, \emptyset) \in S_p \vee T_p$. Thus $\Sigma \subseteq S_p \vee T_p$. Then for any $s_q \in P_\pi$, $q \neq p$, conditions C_3 yield that $s_q \in S_p \vee T_p$. We have shown that $S_p \vee T_p = 1_L$.

To prove (iii), let us suppose that p and q are maximal chains in S^\smile , $\sigma \in p \cap q$, $\sigma 0 \in p \setminus q$ and $\sigma 1 \in q \setminus p$. Now if $(s, t) \in \Sigma$ and $t \not\geq \sigma$ then $t \notin p \cap q$, hence $(s, t) \in S_p \cup S_q$. Thus $S_\sigma \cap \Sigma \subseteq S_p \vee S_q$. Next, it follows from Lemma 19 (2) that $S_\sigma \cap P_\pi \subseteq S_p \vee S_q$. We have now shown that $S_\sigma \leq S_p \vee S_q$. Since it is clear that $S_p \cup S_q \subseteq S_\sigma$, we conclude that $S_p \vee S_q = S_\sigma$. That $S_p \wedge S_q \leq \Sigma$ is obvious.

In proving (iv), we first show that there are no join-prime elements. Suppose, to the contrary, that $J \in L$ is join-prime in \mathbf{L} . Choose any $p, q \in \Pi$ with $0 \in p$ and $1 \in q$ where $0, 1 \prec \emptyset$ in S^\smile . Now it follows from (iii) that $\Sigma \vee S_p \vee S_q = 1_L$. Thus, if $J \not\leq \Sigma$ then it easily follows that either $J \leq S_p$ for every $p \in \Pi$ with $0 \in p$, or else $J \leq S_q$ for every $q \in \Pi$ with $1 \in q$; but then $J \leq S_r \wedge S_{r'}$ for some $r \neq r'$, so in all cases, we conclude that $J \leq \Sigma = \downarrow(\emptyset, \emptyset)$. Now if $s \in S^\smile$ and $J \leq \downarrow(s, \emptyset)$, then since

$$\downarrow(s, \emptyset) = \downarrow(s0, \emptyset) \vee \downarrow(s1, \emptyset),$$

(a consequence of conditions C_0), either $J \leq \downarrow(s0, \emptyset)$ or $J \leq \downarrow(s1, \emptyset)$. Obviously, it follows that there is some $p \in \Pi$ with

$$J \leq \bigwedge_{s \in p} \downarrow(s, \emptyset) = 0_L.$$

However, 0_L by definition is not join-prime. This is a contradiction.

Next, assume that $J \in L$ is meet-prime. First we show that $J \supseteq \Sigma$. Suppose not. Then $(\emptyset, \emptyset) \notin J$. Whenever $s \in S^\smile$ and $(s, \emptyset) \notin J$, then also either $(s0, \emptyset)$ or $(s1, \emptyset)$ lies outside J (by conditions C_0). Thus we have a path $p \in \Pi$ with $(s, \emptyset) \notin J$ whenever $s \in p$. Since $T_p \vee S_p = 1_L$ (by (ii)), then either $T_p \not\leq J$ or $S_p \not\leq J$. Now choose $u = (i, \emptyset)$ with $i \prec \emptyset$, $i \in p$. Since $\downarrow u \not\leq J$, and J is meet-prime, either $T_p \cap \downarrow u \not\leq J$ or $S_p \cap \downarrow u \not\leq J$. Observe that $S_p \cap \downarrow u \subseteq T_p \cap \downarrow u$. Thus in both cases, we have some $x = (t, r) \in T_p \setminus J$. Here $t \notin p$ and so there is some $s \in p$ with $S^\smile \models (\downarrow s) \cap (\downarrow t) = \emptyset$. Then $[\downarrow(s, \emptyset)] \cap [\downarrow(t, r)] = \emptyset = 0_L$. Since we have both $\downarrow(s, \emptyset) \not\leq J$ and $\downarrow(t, r) \not\leq J$, here is a contradiction to our assumption that J is meet-prime. We conclude that $\Sigma \subseteq J$.

Now it follows from (iii) that for $\{p, p'\} \subseteq \Pi$, $p \neq p'$, we have $S_p \leq J$ or $S_{p'} \leq J$. Thus we certainly have that $S_p \subseteq J$ for some p . For such a p ,

$$1_L = S_p \vee T_p \subseteq J.$$

However 1_L is, by definition, not meet-prime, so we have a contradiction, and that finishes our proof of (iv). \blacksquare

LEMMA 21. *Let $Y \subseteq P$, $Y \not\subseteq \Sigma$, and choose any $p \in \Pi$ with $s_p \in Y$. If $Y \not\subseteq S_p$ then choose any $x_0 = (t, r) \in \{(\downarrow Y) \cap \Sigma\} \setminus S_p$ with $n(x_0) = |r|$ the minimum for all elements of $\{(\downarrow Y) \cap \Sigma\} \setminus S_p$.*

(i) *If $Y \subseteq S_p$ then $\overline{Y} = S_p = \overline{\{s_p\}}$.*

(ii) *If $Y \not\subseteq S_p$, then $\overline{Y} = \overline{\{s_p, x_0\}}$. In this case, if $n(x_0) = 0$ then $\overline{Y} = 1_L$, while if $n(x_0) > 0$ then, letting σ be the member of p with $|\sigma| = n(x_0) - 1$, we have that $\overline{Y} = S_\sigma$.*

PROOF. Statement (i) being obvious, we strive to prove (ii). First, assuming that $n(x_0) = 0$, we show that $\overline{\{s_p, x_0\}} = 1_L$. Let i be the unique sequence of length one off of p , and $x_p = (i, \emptyset)$. Now $(i, i) < s_p$ and $(x_p, (i, i), x_0)$ is a condition in C_1 , hence $x_p \in \overline{\{s_p, x_0\}}$. Next, $((\emptyset, \emptyset), x_p, s_p)$ is a condition in C_2 , hence $(\emptyset, \emptyset) \in \overline{\{s_p, x_0\}}$. Thus $\Sigma \subseteq \overline{\{s_p, x_0\}}$. Now $S_p \vee \Sigma \supseteq S_p \vee T_p = 1_L$ (the last equality is by Lemma 20(ii)).

Now we assume that $n(x_0) \geq 1$. It is easily verified that in this case, $Y \subseteq S_\sigma$. Thus, we need to show that in this case, $\overline{\{s_p, x_0\}} = S_\sigma$. For the sake of notational convenience, we now assume that $\sigma 0 \in p$ (instead of $\sigma 1 \in p$), and $\sigma 0 1 \notin p$ (instead of $\sigma 0 0 \notin p$). We have that $x_0 = (t, r)$ with $r \in p$ and $|r| = |\sigma 0|$, hence $r = \sigma 0$. Now $(\sigma 0 1, \sigma 0 1) < s_p$ and $((\sigma 0 1, \sigma 0), (\sigma 0 1, \sigma 0 1), x_0)$ is a condition in C_1 , hence $(\sigma 0 1, \sigma 0) \in \overline{\{s_p, x_0\}}$. Next, $((\sigma 0, \sigma 0), (\sigma 0 1, \sigma 0), s_p)$ is a condition in C_2 , hence $(\sigma 0, \sigma 0) \in \overline{\{s_p, x_0\}}$. Letting $q \in \Pi$ with $\sigma 1 \in q$, we see that $(s_q, s_p, (\sigma 0, \sigma 0))$ is a condition in C_3 , so it follows that $s_q \in \overline{\{s_p, x_0\}}$. Finally, Lemma 20 (iii) gives that $\overline{\{s_p, s_q\}} = S_\sigma$. Thus we have proved that $\overline{\{s_p, x_0\}} = S_\sigma$ and this ends our proof. \blacksquare

LEMMA 22.

(1) *Suppose that $J \in L$, $J \not\subseteq \Sigma$ and $J \neq 1_L$. Then $J = S_p$ for some $p \in \Pi$ or $J = S_\sigma$ for some $\sigma \in S^\omega$. Thus*

$$L = \{1_L\} \cup (\downarrow \Sigma) \cup \{S_p : p \in \Pi\} \cup \{S_\sigma : \sigma \in S^\omega\}.$$

(2) $|\mathbf{L}| = c$ (the continuum).

(3) Every member of L outside the interval $\Sigma/0_L$ is a compact element of \mathbf{L} .

PROOF. Statements (1) and (3) follows from Lemma 21. Statement (2) is then a consequence of the fact that $|P_\pi| = c$ and $\downarrow \Sigma$ is a set of subsets of the countable set Σ . \blacksquare

LEMMA 23. \mathbf{L} is meet semi-distributive.

PROOF. We assume that $J, J_0, J_1 \in L$ and $J \wedge J_0 = J \wedge J_1 = M$. We need to prove that $J \wedge (J_0 \vee J_1) = M$. So let $\alpha \in J \wedge (J_0 \vee J_1)$.

Suppose first that $J_0 \cup J_1 \subseteq \Sigma$ so that $\alpha \in \Sigma$. Then by Lemma 19(4), there is a finite set

$$Q \subseteq (\downarrow [J_0 \cup J_1])^\triangleright \cap \downarrow \alpha \subseteq (J_0 \cup J_1)^\triangleright \cap J$$

with $\alpha \in Q^\dagger$. We have $Q \subseteq J$ and we need to show that $Q \subseteq M$. So let $x \in Q$. Then there are $y, z \in J_0 \cup J_1$ so that $y \sim_0 x$ and $z \triangleright x$. We can assume that $y \leq x$; indeed, if $x = (s, t)$, we can replace y by (s, s) . Then $y \in J \cap J_{i_0} \subseteq M$ ($i_0 = 0$ or $i_0 = 1$). Now by Lemma 19(2), $x \in \overline{\{y, z\}}$, and also $y \in M$ and $z \in J_{i_1}$ ($i_1 = 0$ or $i_1 = 1$). Thus $\{y, z\} \subseteq J_{i_1}$, giving $x \in J \cap \overline{\{y, z\}} \subseteq J \cap J_{i_1}$; and $x \in M$ as desired. This concludes our proof in the case $J_0 \cup J_1 \subseteq \Sigma$.

In the arguments below, we will use several times the elementary fact that if $x \in J$ and $x \in \overline{\{y, z\}}$ with $\{y, z\} \subseteq J_0 \cup J_1$ and $y \leq x$, then $x \in M$. The proof of this elementary fact is contained the final three sentences of the last paragraph.

Now assume that $J_0 \cup J_1 \not\subseteq \Sigma$. As in Lemma 21, we choose any $s_p \in J_0 \cup J_1$ with $p \in \Pi$. If $J_0 \cup J_1 \subseteq S_p$, then $J_0 \vee J_1 = S_p = \overline{\{s_p\}}$ and this is equal to J_i where $s_p \in J_i$, thus this case is trivial. Hence we assume further that $J_0 \cup J_1 \not\subseteq S_p$. Following Lemma 21, we choose $x_0 = (t, r) \in J_0 \cup J_1$ with $r \in p$ so that

$$J_0 \vee J_1 = \overline{J_0 \cup J_1} = \overline{\{s_p, x_0\}};$$

and if $n(x_0) = |r| = 0$ then $J_0 \vee J_1 = 1_L$, while if $n(x_0) > 0$ then we take σ to be the unique sequence with $r \prec \sigma$ and we are guaranteed that $J_0 \vee J_1 = S_\sigma$ in this case.

We next show that every element of $J \cap (J_0 \vee J_1) \cap \Sigma$ belongs to M . Suppose that $\alpha \in J \cap (J_0 \vee J_1) \cap \Sigma$. Say $\alpha = (u, v)$. Here we have $v \leq r$ since $\alpha \in \{s_p, x_0\}$. If $u \notin p$ then $(u, u) \leq \alpha$ and $(u, u) < s_p$ hence $(u, u) \in M$. Since $(u, v) \in \overline{\{(u, u), (t, r)\}}$ by C_1 (because $v \leq r$), then $\alpha \in J \cap (J_0 \cup J_1) \subseteq M$. If $u \in p$ then we actually must have $u \leq r$. In this case, let $u' \prec u$, $u' \notin p$. Then $(u', u') < \alpha$ and $(u', u') \in M$ by what we just proved. Then $(u', v) \in \overline{\{(u', u'), x_0\}}$ by C_1 ; since $(u', v) < \alpha$ then $(u', v) \in M$. Finally, $(u, v) = \alpha \in \overline{\{(u', v), s_p\}}$ by C_2 , so we get $\alpha \in M$. Thus in all cases, $\alpha \in M$ if $\alpha \in \Sigma$.

Finally, suppose that $\alpha = s_q$, $q \in \Pi$. We assume that $q \neq p$. Choose $u \in p \cap q$ with $ui \prec u$ and $ui \in p \setminus q$. Now $(ui, ui) < \alpha$ and so by what we have just proved in the last paragraph, $(ui, ui) \in M$. Also, $s_q = \alpha \in \overline{\{(ui, ui), s_p\}}$ by C_3 . Thus in this final case also, we get $\alpha \in M$. ■

THEOREM 24. *The lattice \mathbf{L} is algebraic, meet semi-distributive, has compact 1, is of cardinality 2^{\aleph_0} , and has no meet-prime elements and no join-prime elements.*

PROOF. This summarizes Lemma 20(iv), Lemma 22 (2), (3) and Lemma 23. ■

COROLLARY 25. *There exists a lattice that possesses all known properties of lattices of sub-quasivarieties except JKP.*

PROOF. Let \mathbf{M} be a lattice obtained by adding a new 0 to the lattice dual to \mathbf{L} of Theorem 24, so that \mathbf{M} has a smallest non-zero element b and the filter $\uparrow b$ is isomorphic to the dual of \mathbf{L} . Then \mathbf{M} is join semi-distributive, atomic (every element contains an atom) and dually algebraic. Also, \mathbf{M} supports an equa-closure operator h . We can just define $h(x) = 1$ for $x \neq 0$ and $h(0) = 0$. On other hand, \mathbf{M} has no join-prime element other than b , thus JKP fails. ■

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KIRA ADARICHEVA
Department of Mathematics
Harold Washington College
Chicago, IL 60601 USA
ki13ra@yahoo.com

MIKLOS MARÓTI
University of Szeged
Bolyai Institute
Aradi Vertanuk Tere 1
H6720 Szeged, Hungary
mmaroti@math.u-szeged.hu

RALPH MCKENZIE
Department of Mathematics
1326 Stevenson Center
Vanderbilt University
Nashville, Tennessee 37240 USA
mckenzie@math.vanderbilt.edu

JB NATION
Department of Mathematics
University of Hawaii
Honolulu, HI 96822 USA
jb@math.hawaii.edu

ERIC R. ZENK
Department of Mathematics
1326 Stevenson Center
Vanderbilt University
Nashville, Tennessee 37240 USA
eric.zenk@vanderbilt.edu